Topology from Fermions

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We show that there can be an odd number of fermionic zero modes in certain string-like, non-topological, bosonic configurations while the number of zero modes in the vacuum vanishes. We use the index $n_L - n_R$, where n_L and n_R are the number of left- and right-moving zero modes, to argue that such non-topological bosonic field configurations cannot evolve continuously into the vacuum and that they lie in a distinct topological sector. This topology is not given by the homotopy groups of the bosonic vacuum manifold; instead it is a property of the fermionic sector of the theory.

It is well appreciated that in bosonic field theories whose vacuum manifold is topologically non-trivial, classical, static solutions of the field equations can exist and be stable. These solutions globally minimize the field energy in the topological sector in which they lie. The fermions in the field theory have so far only played a passive role – their modes are examined in the background of the topological solution and often one or more zero modes are found. Using certain "index theorems", the zero modes can often be directly related to the topological properties of the background. In this sense, the zero modes are an outcome of the background topology.

In this paper we shall be interested in a related but different aspect of field theories. We ask the question: can the bosonic background be topologically trivial and yet the fermions on this background be topologically non-trivial? We answer this question in the affirmative by showing the existence of an odd number of zero mode solutions on topologically trivial backgrounds. We then argue that the presence of the odd number of zero modes signals that the field configuration is topologically distinct from the vacuum and hence cannot relax into it.

The field theory we will consider has been inspired by the standard model of the electroweak interactions and so we will adopt the same notation for the various fields (*eg.* see [1,2]). In addition to the usual standard model fields, we will include a complex Higgs field Δ, transforming in the adjoint representation of $SU(2)$. The full Lagrangian is:

$$\mathcal{L} = \mathcal{L}_{EW} + \mathcal{L}_\Delta + \mathcal{L}_f \qquad (1)$$

where,

$$\mathcal{L}_{EW} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \Phi|^2 + V(\Phi) \qquad (2)$$

$$\mathcal{L}_\Delta = |D_\mu \Delta|^2 + U(\Delta, \Phi) \qquad (3)$$

$$\mathcal{L}_f = \quad i\bar{\Psi}\gamma^\mu D_\mu \Psi + i\overline{e_R}\gamma^\mu D_\mu e_R + i\overline{\nu_R}\gamma^\mu \partial_\mu \nu_R +$$

$$
\begin{aligned}
&- [h'\overline{e_R}\Phi^\dagger\Psi + h\overline{\nu_R}\left(\Phi^T(i\tau_2)^\dagger\Psi\right) + \\
&+ \frac{1}{2}h''\bar{\Psi}i\tau_2\Delta\Psi^c + \frac{1}{2}\overline{\nu_R}M_R(\nu_R)^c + h.c.]
\end{aligned} \tag{4}
$$

where $U(\Delta, \Phi)$ and $V(\Phi)$ are general quartic potentials. The covariant derivatives are defined as:

$$
\begin{aligned}
D_\mu\Psi &= \left(\partial_\mu - i\frac{g}{2}\tau^a W^a_\mu + i\frac{g'}{2}Y_\mu\right)\Psi \\
D_\mu e_R &= \left(\partial_\mu + ig'Y_\mu\right)e_R \\
D_\mu\Phi &= \left(\partial_\mu - i\frac{g}{2}\tau^a W^a_\mu - i\frac{g'}{2}Y_\mu\right)\Phi \\
D_\mu\Delta &= \partial_\mu\Delta + ig'Y_\mu\Delta - i\frac{g}{2}\left[\tau^a W^a_\mu, \Delta\right] \ .
\end{aligned} \tag{5}
$$

The fields in the model are defined as: $\Psi = (\nu_L, e_L)^T$, $\nu^c = C\bar{\nu}^T$ and $\Psi^c = C\bar{\Psi}^T$where C is the charge conjugation matrix and τ^a are the Pauli matrices. $W^a_{\mu\nu}$ and $F_{\mu\nu}$ are the field strength tensors for W^a_μ and Y_μ respectively, and the Higgs triplet Δ which transforms according to the adjoint representation of $SU(2)_L$ is

$$
\Delta = \begin{pmatrix} H^+ & H^{++} \\ H^0 & -H^+ \end{pmatrix} \ . \tag{6}
$$

The parameters of the model can be chosen so that Δ and Φ get vacuum expectation values:

$$
h''\Delta = \begin{pmatrix} 0 & 0 \\ M_L & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \tag{7}
$$

with M_L and η being constants. This breaks the original $[SU(2)_L \times U(1)_Y]/Z_2$ symmetry down to $U(1)_{EM}$. The vacuum manifold is an S^3 and contains no incontractable paths. This means that there are no topological string solutions in the model. However, there do exist non-topological string solutions [1] – called the Z-string – and these shall form the basis of the background bosonic configuration that we shall study.

Let us define the Z gauge field by

$$
Z_\mu \equiv \cos\theta_w W^3_\mu - \sin\theta_w Y_\mu \tag{8}
$$

where θ_w is the weak mixing angle ($\tan\theta_w \equiv g'/g$). Then the bosonic field configuration that we choose to study is:

$$
qZ_\theta = -\frac{v(r)}{r} \ , \quad \Phi = \phi\begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \quad h''\Delta = \begin{pmatrix} 0 & 0 \\ M_L & 0 \end{pmatrix} \tag{9}
$$

with

$$
\phi \equiv \eta f(r) e^{i\theta}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}
$$

and all other gauge fields set to zero. Also, $q \equiv \sqrt{g^2 + g'^2}/2$, $v(r)$ and $f(r)$ are profile functions, and (r, θ, z) are cylindrical coordinates. Note that the configuration is independent of the z coordinate.

Without the Δ field, the field equations of motion can be solved and this determines the profile functions $f(r)$ and $v(r)$. The solution is known as the Z-string. It is unstable except for large θ_w and small values of the ratio of scalar (Φ) to Z mass. With the Δ included, the configuration (with f and v as in the Z-string) is not a solution. This can easily be seen since $D_\theta \Delta \propto Z_\theta \Delta \propto 1/r$ even as $r \to \infty$ while the potentials U and V go to zero exponentially fast at large r. The energy of the configuration is logarithmically divergent, reminiscent of the global $U(1)$ string. None of these features really concerns us since we are free to consider fermionic modes on any bosonic background we wish. The fermionic zero modes that we will obtain will be well-localized.

The neutrino and electron equations of motion in the bosonic background are:

$$
\begin{aligned}
i\gamma^\mu D_\mu \nu_L &= h\phi^* \nu_R + M_L (\nu_L)^c \\
i\gamma^\mu \partial_\mu \nu_R &= h\phi \nu_L + M_R (\nu_R)^c \qquad (11) \\
i\gamma^\mu D_\mu e_L &= h' \phi e_R \\
i\gamma^\mu D_\mu e_R &= h' \phi^* e_L \ . \qquad (12)
\end{aligned}
$$

In the standard model, *i.e.* in the absence of Δ, M_L and M_R, these Dirac equations have been solved. The fermionic equations are known to have a well-localized (*i.e.* with an exponential fall off) zero mode just as in the case of topological vortices [3]. There is a zero mode for the electron propagating in one direction along the string and another zero mode for the neutrino propagating in the opposite direction [4,5,1]. In the case of massless neutrinos, *i.e.* $h\eta = M_L = M_R = 0$, the zero mode is not discretely normalizable but is delta-function normalizable. In other words, it is part of a continuum of modes [2]. If we include a massive neutrino with $h\eta \neq 0$, $M_R \neq 0$ and $M_L = 0$ there is one well-localized, discretely normalizable solution. The electron equations in (12) also have one well-localized discrete zero mode regardless of the values of M_L and M_R.

Since the Z-string configuration is non-topological, it is interesting to study the fate of the zero modes when the background is perturbed. It has been shown that the perturbations in the Higgs and gauge sectors continuously mix the electron and neutrino zero modes (see Fig. 1) and convert the zero modes into two low lying massive states [6,7]. In this process, the number of eigenmodes is conserved, only the initially vanishing eigenvalues have flowed to non-vanishing values.

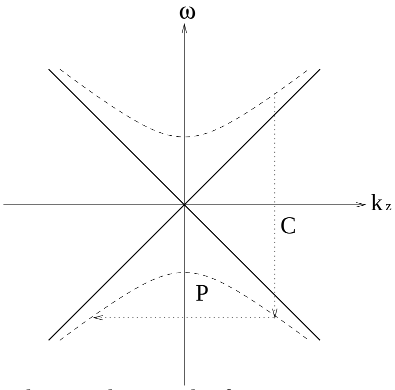

FIG. 1. In the usual case, the fermionic spectrum contains two lines (dashed) that describe zero modes boosted in the z-direction. One line describes the left-moving and the other the right-moving zero modes. Perturbations of the bosonic background can mix the two zero modes at the origin and convert them into two massive modes that lie on hyperbolae. The points on the two hyperbolae are related by CP transformations as shown.

Now we study neutrinos with the general mass matrix:

$$\mathcal{M} = \begin{pmatrix} M_L & M_D \\ M_D & M_R \end{pmatrix} \tag{13}$$

where the Dirac mass $M_D = h\eta f(r)e^{i\theta}$. The electron still has a zero mode since its equations are decoupled from those of the neutrino. Now we shall now prove that if M_L and M_R are large enough compared to M_D, then there is no neutrino zero mode on the string.

Let us write the neutrino Dirac equations as:

$$i\gamma^\mu \mathbf{D}_\mu \Psi = \mathbf{M}\Psi \tag{14}$$

where

$$\Psi = \begin{pmatrix} \nu_L \\ \nu_R^c \\ \nu_L^c \\ \nu_R \end{pmatrix} \quad , \quad \mathbf{M} = \begin{pmatrix} 0 & 0 & M_L & M_D \\ 0 & 0 & M_D & M_R \\ M_L^\dagger & M_D^\dagger & 0 & 0 \\ M_D^\dagger & M_R^\dagger & 0 & 0 \end{pmatrix} \tag{15}$$

and $\mathbf{D}$ is matrix valued so as to recover the original Dirac equations (12). We can get some information about the spectrum of the Dirac operator by squaring it. If we take $\Psi(\vec{x},t) = e^{i\omega t}\psi(\vec{x})$ with the normalization $\int d^2x\, \psi^\dagger\psi = 1$, we get

$$\omega^2 = -\int d^2x\, \psi^\dagger \mathbf{D}^2 \psi + \int d^2x\, \psi^\dagger[\sigma^{\nu\mu}F_{\mu\nu} + |M_D|^2]\psi$$
$$+4\int d^2x\, (T[\alpha,\delta] + T[\beta,\gamma])$$

where the contribution of the Majorana masses is all in the functional T:

$$\begin{aligned} T[\alpha,\delta] \equiv\ & |M_L|^2|\alpha|^2 + |M_R|^2|\delta|^2 + \\ & (M_L M_D^\dagger + M_D M_R^\dagger)\alpha\delta + \\ & (M_L^\dagger M_D + M_D^\dagger M_R)\alpha^*\delta^* \end{aligned} \tag{16}$$

and

$$\nu_L = \begin{pmatrix} \alpha \\ \beta \\ -\alpha \\ -\beta \end{pmatrix} \ , \ \ \nu_R = \begin{pmatrix} \gamma \\ \delta \\ \gamma \\ \delta \end{pmatrix} \ . \tag{17}$$

If the Majorana masses vanish, then one sets $T = 0$. Here we know for sure that there is a zero mode *i.e.* a choice of ψ such that $\omega^2 = 0$. So let us write:

$$\omega^2 = \Omega[\psi] + 4 \int d^2x (T[\alpha, \delta] + T[\beta, \gamma]) \tag{18}$$

From the zero Majorana mass case, we know that the minimum value of the functional Ω is zero. That is, no matter what ψ we choose, we cannot make Ω negative. The least we can make it is zero. Then if we can prove that T is positive (and non-vanishing) for all ψ, we will have shown that $\omega^2 > 0$ and that there is no neutrino zero mode.

It is not hard to see that $T[\alpha, \delta] > 0$ for any choice of α and δ provided

$$|M_L M_R| > \max_{(r,\theta)} |M_L M_D^\dagger + M_D M_R^\dagger| \tag{19}$$

Note that M_D is a function of (r, θ) and we would like to find the maximum value of the expression on the right-hand side. The inequality can be simplified further by using the form of M_D given below eq. (13). Then we find that the condition for $T > 0$ is:

$$\frac{|M_L||M_R|}{|M_L| + |M_R|} > h\eta \ . \tag{20}$$

One can check that if this condition is saturated, then it is possible to find α and δ such that $T = 0$ at some spatial points. Therefore the condition in eq. (19) is equivalent to having $T[\alpha, \delta] > 0$ for all α, δ, r and θ. Hence if the parameters satisfy equation (20), then there are no neutrino zero modes.

The possibility of not having a neutrino zero mode while the corresponding electron zero mode is still present is important from the point of view of the string stability under perturbations. In this case, perturbations of the string cannot lift the electron zero mode into a massive mode. A single zero mode (see Fig. 2) cannot be continuously deformed as in the case described by Fig. 1. The central point $(\omega, k) = (0, 0)$ cannot be moved up or down without violating CP invariance*. For non-topological strings this conclusion seems paradoxical because there is no topology in the bosonic sector that prevents the string from decaying into the vacuum. Yet there is no electron

*It is easy to check that the bosonic background in eq. (9) remains invariant under CP transformations.

zero mode in the vacuum. The only resolution seems to be that the unpaired electron zero mode provides topological stability to the string and that the topology enters via the fermionic sector of the model.

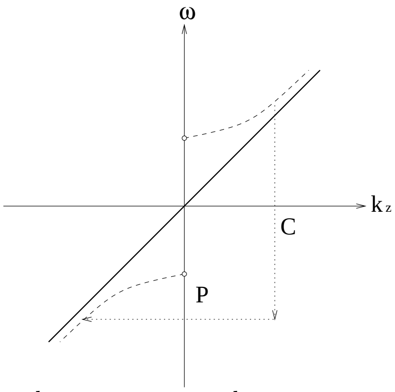

FIG. 2. In the present case, the spectrum contains only the line of boosted electron zero modes (the $\omega = +k_z$ line). Perturbations of the background cannot change the line into the spectrum for massive fermions (hyperbolae). In particular, perturbations cannot convert the single zero mode ($\omega = 0 = k$) into a massive mode because CP invariance would require it to split into two modes, leading to a discontinuous spectrum.

While we do not know if there is a topological charge density that can be associated with this "fermionic topology", there are certainly integers, namely the number of left- and right- moving fermionic zero modes (n_L and n_R), that characterizes the field configuration. We have argued that $n_L - n_R$ plays the role of the topological charge and cannot change under continuous evolution of the fields. Since $n_L - n_R$ in the vacuum is zero and in our chosen background is 1, the background and the vacuum are in distinct topological sectors.

The divergence in the energy associated with the constant Δ configuration is not relevant for the above discussion since it arises from the long distance behavior of the bosonic fields, while the spectrum of fermionic zero modes is set principally by the behavior of the fields near the string core – the electron zero mode wave function falls off exponentially fast as we move away from the string core. Also, following the example of global cosmic strings, it is possible to construct a modified configuration so as to make the total energy finite. Consider the configuration in eq. (9) together with its anticonfiguration *i.e.* $\theta \to -\theta$ in (10). Together the bosonic configuration and anti-configuration have finite cross-sectional energy since the energy density in the fields falls off as in a two-dimensional dipole, $\mathcal{H} \sim d^2/r^4$, where d is the separation of the configuration and anticonfiguration. Now there will be an up-moving electron zero mode on the configuration and a down-moving electron zero mode on the anti-configuration. So the total configuration has net trivial topology. However, for the configuration to evolve into the vacuum, the configuration and anti-configuration have to overlap so that the two electron zero modes can mix. This is closely parallel to the case of global strings where the energy of a single string is also logarithmically divergent. One can consider a widely separated string and anti-string configuration which will have finite cross-sectional energy but be topologically trivial. The decay of the system into the vacuum can only be achieved by bringing the string and anti-string together so that they can annihilate.

In conclusion, we have shown that certain topologically trivial bosonic field configurations cannot evolve into the vacuum due to the topological properties of the fermions in the model. The standard model with massive neutrinos is a convenient example for this phenomenon. We have argued that the existence of the electron zero mode and the absence of a neutrino zero mode makes the Z-string-like configuration topologically distinct from the vacuum. We have not determined the ground state of the fields in this non-trivial topological sector since this can only be found by solving the full equations of motion in which the bosonic fields also feel the back-reaction of the fermionic zero modes.

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